14. Falling Tower

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Contents

1 Problem Statement

Identical discs are stacked one on top of another to form a freestanding tower. The bottom disc can be removed by applying a sudden horizontal force such that the rest of the tower will drop down onto the surface and the tower remains standing. Investigate the phenomenon and determine the conditions that allow the tower to remain standing.

2 Introduction

Our work examines a very simple process – finding out whether a tower made from discs will fall or remain intact when the bottom disk is removed. This problem can be described using only relatively simple physics, yet creating a model that would give satisfying results requires an in-depth analysis of the phenomena.

We first start by providing a thorough description of the phenomena and qualitative model. After that in the quantitative part, the model for describing the tower is presented which is later compared to the results of two experiments.

2.1 Disc

We define a disc to be a homogeneous cylinder whose diameter is bigger than its height.

We will denote its diameter as d, its height as h and its mass as m.

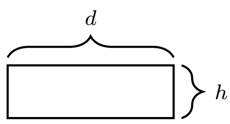


Figure 1: Disc

We will use μ as the friction coefficient for friction between two discs and μ^* as the friction coefficient between a disc and the underlay.

3 Qualitative analysis

We will draw the situation as a vertical cross-section that goes through the centers of the discs and contains the vector of the pulling force F. The other relevant forces present in the initial moment are:

- The gravity force of each disc $m \cdot g$. These forces add up from the top, so the total gravity force on the platform is $n \cdot m \cdot g$.
- The static dry friction force between the bottom disc and the platform F_f^*, going against the pulling force F. Since the normal force is the weight force on the platform, this friction force can't exceed $\mu_s^* \cdot n \cdot m \cdot g$, where μ_s^* is the static dry friction coefficient between the platform and disc materials. If the pulling force is not big enough to counteract this force, the discs won't move at all.

- Between each pair of touching discs, there is a static dry friction force F_{fi}, where i is the number of discs above the lower disc. The friction works against the movement of the bottom disc and, as a reaction, acts on the top disc in the same direction. Since the normal force is the weight force from the discs above, the friction force can't exceed $\mu_s \cdot (n-i)mg$, where μ_s is the static dry friction coefficient between the discs.

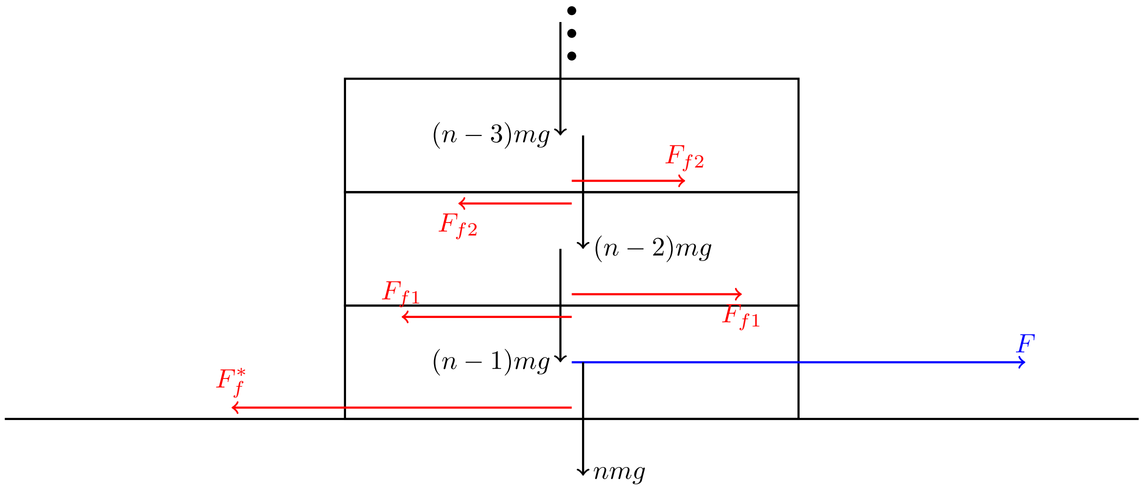


Figure 2: Forces at the beginning

3.1 Greater forces

In this subsection we will discuss what happens to the tower when the forces are great enough – i.e. when the tower does not move as a whole. The forces that are lower will be discussed in the next section.

We can describe the resultant forces in each disk, which touches the disk below. Assume there are i discs stacked above this one.

The force pushing this disc forward is then:

$$F = \mu_k \cdot m \cdot g \cdot (i+1) - \mu_k \cdot m \cdot g \cdot i$$

$$F = \mu_k \cdot m \cdot g$$

Which is independent of the number of discs stacked above this particular one.

According to the Laws of motion, for the acceleration $a = \frac{F}{m}$, all the disc when touching the disc below will horizontally accelerate with the same acceleration.

$$a = \mu_k \cdot g$$

This is however not true for the angular acceleration. The torque and therefore the angular acceleration depends on the angle of the two neighbouring discs. The rotation of these is then what makes the tower fall.

Another interesting factor is that all these forces are independent of the applied force. Therefore the overall rotation of the disc only depends on the time of contact with the bottom disc. As the time of contact is lower with increasing force, we can straight away say that, as the force has no vertical component, with increasing forces the tower should stand more easily.

Let's now take one step back to the very beginning of the whole process. There the angles equal therefore all the torques are equal and so are the angular velocities.

It may seem as if all the disc move accelerate all the same, their angles and velocities should stay the same. This is in general case not true as at any point in time due to their rotation and movement the contact between any two neighbouring discs. This would cause the lower of these to accelerate faster until it hits its neighbour which causes it to decelerate and its neighbour to accelerate. The whole process starts showing signs of chaotic behavior.

3.2 Lower forces

In this section we will cover the case when the tower moves as a whole and we will show how small the applied force must be for this to happen. We will cover the constant forces first and then we will show how this can be extended to an arbitrary function of force.

As already discussed, in order for the system to start moving, F has to be greater than $\mu_\mathrm{s}^* n \cdot m \cdot g$. Otherwise the applied force and the friction with ground will cancel out and the bottom disc will not accelerate and the tower will not move.

When the forces are higher the tower starts to move. We then have the force $F - \mu_\mathrm{s}^* n \cdot m \cdot g$

The friction between the two discs is at most $\mu_\mathrm{s}(n-1) \cdot m \cdot g$. If the force cannot surpass this value, the tower will move as a whole.

However even if the force is higher than the some of the two forces mentioned, the entire tower might still move. This is due to the fact that all the discs except for the bottom one are accelerated with force which is at most $\mu_\mathrm{k} \cdot m \cdot g$, therefore, if the resultant force acting on the bottom disc is not higher than this, all the disc will accelerate with the same acceleration.

Therefore if

$$F \leq \mu_\mathrm{s}^* n \cdot m \cdot g + \mu_\mathrm{s} n \cdot m \cdot g$$

the tower either does not move or moves as a whole.

Lets now take a brief look when the forces are not constant.

When the force is lower than $(\mu_\mathrm{s}^* + \mu_\mathrm{s}) \cdot n \cdot m \cdot g$ throughout its domain, the situation is the same as for the constant force – the tower bottom disc never slides out.

As for the constant force the disc always slides out when the force is always higher than the sum of friction forces maximums.

What is different is when the force is higher only in part of the domain. Then the disk gets accelerated but may or may not slide out before the friction slows it down. Both of these options are possible and it depends only on what the function precisely looks like.

4 Quantitative model

As the problem involves many bodies and their behavior is rather complex as discussed in the previous section, in order to solve it, we had to simplify the problem a little.

The whole process can be split into two parts – the process of removing the bottom disc when the bottom disc touches the discs above, and the stabilization of the tower when they are not touching. We will simplify the first part and assume that all the discs except for the bottom one act as glued together.

This may seem as a strong simplification, however we will briefly show why this it not as strong as it may seem and moreover does not change the essence of the problem.

Imagine the process of rotation of the two neighbouring discs as described in the qualitative section. These sometimes do not touch which causes the increase in the angular velocity of the bottom one. However at some moment, the bottom one will touch the disc above, which will cause it decelerate and the angles to equalize. Therefore taking look at their angles at bigger time intervals, they will both have almost the same angle and therefore appear to have the same angular acceleration.

Our simplified model does resemble this behaviour as the discs in the cylinder rotate together.

Another implication of our simplified model is that the disc on the top travelled smaller distance than the discs on the bottom. As it may seem incorrect from the fact that the acceleration is everywhere the same, this behavior occurs in reality.

Remember that the acceleration was all the same only if all the disc below the given disk touched the bottom disk, otherwise it was zero. If we assign some probability (lower than one) that the two discs touch at given time, we can see that with increasing number of discs beneath the given disc the probability decreases and therefore the discs horizontally lacks behind the ones below more and more. In reality this effect would be bit lowered by interactions between two discs both separated from the bottom one.

Both these factors were also observed experimentally. In our experiments however, it happened that the moment of force on the bottom discs (not meant the ultimately bottom one) was so great, that the bottom of the tower shattered. Although this behaviour is not predicted by our model, as for so big friction forces the tower would have fallen anyway even if such behaviour was restricted, therefore our model should predict these cases correctly as fallen.

4.1 The motion of the bottom disc

Assume the bottom disc is pushed in one direction by a general force[1] $F(t, s_{\mathrm{d}})$.

Due to the friction there will be two friction forces, one with the rest of tower and second with the desk on which the disc slides, both acting in the opposite direction.

For the friction with the desk:

$$F_f^* = \mu_k^* \cdot n \cdot m \cdot g$$

[1] This force could also depend on other parameters and the model would work, but time and distance travelled are the most reasonable factors.

Where the μ_k^* is the kinematic friction coefficient with desk.

Similarly for the friction with discs:

$$F_f = \mu_k \cdot (n-1) \cdot m \cdot g$$

The μ_k being the kinematic friction coefficient between the discs.

Subtracting these from the applied force and linking it to the acceleration using the Laws of motion, we get:

$$\ddot{s}_{\mathrm{d}}(t, s_{\mathrm{d}}) = \frac{F(t, s_{\mathrm{d}})}{m} - \mu_k^* \cdot n \cdot g - \mu_k \cdot (n-1) \cdot g$$

Depending on the form of the function depicting the force, this may or may not have an analytical solution. For the two functions of force we measured in our experiments, the constant and the force linearly depending on the travelled position, there was an analytical solution.

For the constant function $F(t, s_{\mathrm{d}}) = F$:

$$\ddot{s}_{\mathrm{d}}(t, s_{\mathrm{d}}) = \frac{F}{m} - \mu_{\mathrm{k}}^* \cdot n \cdot g - \mu_{\mathrm{k}} \cdot (n-1) \cdot g$$

$$s_{\mathrm{d}} = \frac{t^2}{2} \left(\frac{F}{m} - \mu_{\mathrm{k}}^* \cdot n \cdot g - \mu_{\mathrm{k}} \cdot (n-1) \cdot g \right)$$

And for the force linear to the distance $F(t, s_{\mathrm{d}}) = F_0 - C \cdot s_{\mathrm{d}}$, where C is some positive constant:

$$\ddot{s}_{\mathrm{d}} + \frac{C \cdot s_{\mathrm{d}}}{m} = \frac{F_0}{m} - \mu_{\mathrm{k}}^* \cdot n \cdot g - \mu_{\mathrm{k}} \cdot (n-1) \cdot g$$

This is a non-homogeneous second order differential equation. As a solution we can take the sum of the solution to the homogeneous version of the equation and any partial solution. Doing so[2] we obtain the following general solution:

$$s_{\mathrm{d}} = c_1 \sin\left(\sqrt{\frac{C}{m}} t \right) + c_2 \cos\left(\sqrt{\frac{C}{m}} t \right) + \frac{m}{C} \left(\frac{F_0}{m} - \mu_{\mathrm{k}}^* \cdot n \cdot g - \mu_{\mathrm{k}} \cdot (n-1) \cdot g \right)$$

Although this predicts oscillatory behavior, as the applied force in our experiment was throughout the motion positive, there will be no oscillations.

After setting the initial conditions to be zero at time zero ($s_{\mathrm{d}}(0) = 0$, $\dot{s}_{\mathrm{d}}(0) = 0$), we get:

$$s_{\mathrm{d}}(t) = \frac{m}{C} \left(1 - \cos\left(\sqrt{\frac{C}{m}} t \right) \right) \left(\frac{F_0}{m} - \mu_{\mathrm{k}}^* \cdot n \cdot g - \mu_{\mathrm{k}} \cdot (n-1) \cdot g \right)$$

We obtained the equation describing the location of the bottom disc throughout the process.

4.2 Motion of the cylinder

We shall first define some parameters describing the cylinder. $L = (n-1)h$ will be the height of the cylinder, its diameter will be the same as that of the disc (d) and its mass will be $M = (n-1)m$.

[2] Note that as the coefficient is always positive, the solution to the characteristic equation will be two complex conjugates. As is standard, we will take the real and the imaginary part of the solution as two independent solutions.

We will first describe the horizontal distance of the center of from its initial position and its rotation.

4.2.1 Horizontal position of the cylinder

The only horizontal force acting on the cylinder is the reaction to the friction force. Therefore we use the Laws of motion to describe the horizontal acceleration of the center of mass of the cylinder.

$$\ddot{s}_{\mathrm{c}} = \frac{\mu_{\mathrm{k}} M g}{M}$$

$$\ddot{s}_{\mathrm{c}} = \mu_{\mathrm{k}} g$$

$$s(t) = \frac{1}{2}\mu_{\mathrm{k}} \cdot g \cdot t^2$$

Which tells us the position of the cylinder in the given time.

4.2.2 Rotation of the cylinder

There are three forces acting on the cylinder – the reaction to the friction, the gravity force and the reaction to the gravity force.

As the gravity acts in the center of mass, the torque of this force will be zero. Therefore the resultant torque will be the sum of the torques of the two other forces.

$$\tau = \tau_{\mathrm{f}} + \tau_{\mathrm{r}}$$

Let's denote $\Delta s = s_{\mathrm{d}} - s_{\mathrm{c}}$ the distance between the centers of mass at given time. This can be calculated from the equation derived in previous sections.

As the reaction to the gravity act in the place when the cylinder and the disc touch and the reaction has upwards direction, the perpendicular distance between the center of mass of the cylinder and the force is $\Delta s - \dfrac{d}{2}$ and the torque equals:

$$\tau_{\mathrm{r}} = \left(\Delta s - \frac{d}{2}\right) \cdot M \cdot g$$

To calculate the torque of the friction force, we have to calculate the perpendicular distance first and in order to do this, we have to split the calculation into few cases.

In all three cases, it can be shown[3] that $\xi = |\angle STV| = \arcsin\left(\dfrac{(\Delta s - \frac{d}{2}) \cdot \cos\theta}{p}\right)$ and $|\angle STU| = \arcsin\left(\dfrac{\frac{L}{2} \cdot \cos\theta}{p}\right)$, where $p = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\Delta s - \frac{d}{2}\right)^2 - 2\frac{L}{2}\left|\Delta s - \frac{d}{2}\right| \sin\theta}$.

Therefore the perpendicular distance R to the friction force (i.e. the $|VT|$) equals:

$$R = \left|\Delta s - \frac{d}{2}\right| \cdot \cot\xi$$

[3]For geometric details head to the appendix.

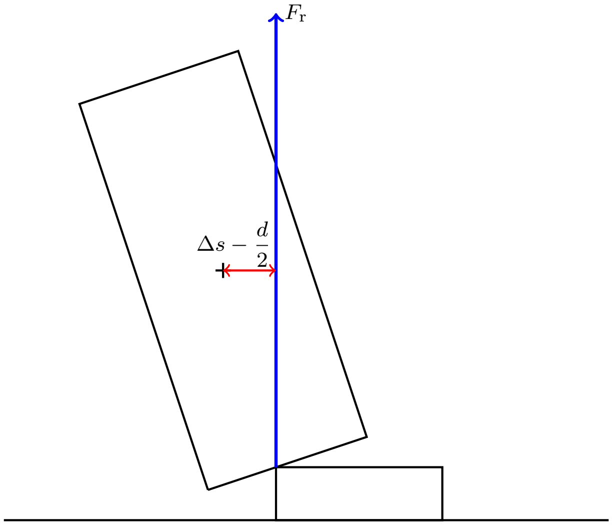


Figure 3: The reaction to the gravity force

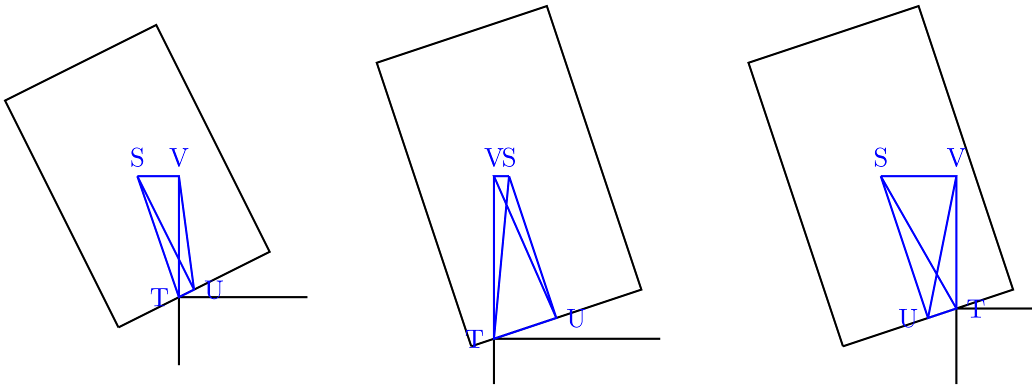


Figure 4: The three possible cases depending on relative horizontal position of S, T and U

Therefore the resultant torque from the reaction to the friction force will equal:

$$\tau_f = R \cdot \mu_k \cdot M \cdot g$$

And the total torque will equal:

$$\tau = \left(\Delta s - \frac{d}{2}\right) \cdot M \cdot g + R \cdot \mu_k \cdot M \cdot g$$

We can link the torque to the angular acceleration using the following equation:

$$\ddot{\theta} = \frac{\tau}{I}$$

This formula works only when the moment of inertia does not change. As the cylinder rotates around its center of mass the whole time, this condition holds.

For a cylinder rotating around its center as ours, the moment of inertia equals[1]:

$$I = \frac{M}{12}\left(\frac{3}{4}d^2 + L^2\right)$$

And the angular acceleration equals:

$$\frac{12((\Delta s - \frac{d}{2}) \cdot g + R \cdot \mu_k \cdot g)}{\left(\frac{3}{4}d^2 + L^2\right)}$$

Which is a nonlinear differential equation describing the rotation of the cylinder.

4.3 Touch termination

In this section we will discus how it is possible to tell, whether the disc and the cylinder stopped touching.

There are two ways this could happen. Either the distance between them is too big or the cylinder starts touching the ground and the discs then immediately slides out.

4.3.1 Termination by ground

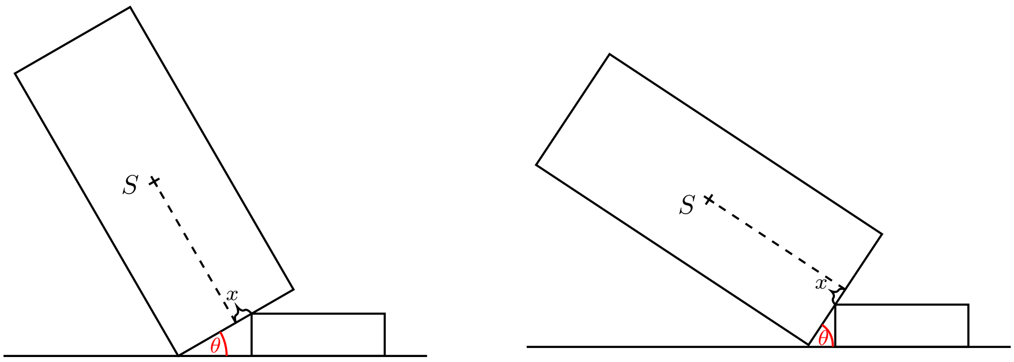


Figure 5: The two possible cases for touching both the ground and the lower disc

$$\left(\Delta s - \frac{d}{2} \geq \frac{l}{2}\sin\theta \wedge \sin\theta \geq \frac{h}{\frac{d}{2}+x}\right) \vee \left(\Delta s - \frac{d}{2} \leq \frac{l}{2}\sin\theta \wedge \sin\theta \geq \frac{h}{\frac{d}{2}-x}\right)$$

where $x = \frac{l}{2}\cot|\angle STU|$. When the cylinder is touching both the ground and the disc, it forms a right triangle with angle θ. We must distinguish two cases depending on whether the center of the cylinder's bottom is above the touching point. Looking back at the three cases shown when determining torque for the friction force, we can see that it's true if the horizontal distance of S and U, which can be calculated as $\frac{L}{2}\sin\theta$, is smaller than $\Delta s - \frac{d}{2}$. Depending on this, the hypotenuse of the triangle is $\frac{d}{2} \pm x$, where x is the distance from the touching point to the center of the cylinder's bottom. That can be easily calculated from the aforementioned angle STU. If the triangle can be formed, that is, $\sin\theta = \frac{h}{\text{hypotenuse}}$, then we know the cylinder is touching the ground.

4.3.2 Termination by distance

The second option how the two discs disconnect is when the two discs are too far away.

The distance between the center of mass of the cylinder and the touching point equals:

$$u = \frac{\sqrt{L^2+d^2}}{2}$$

Therefore the angle α equals:

$$\alpha = \arcsin\left(\frac{L}{2u}\right)$$

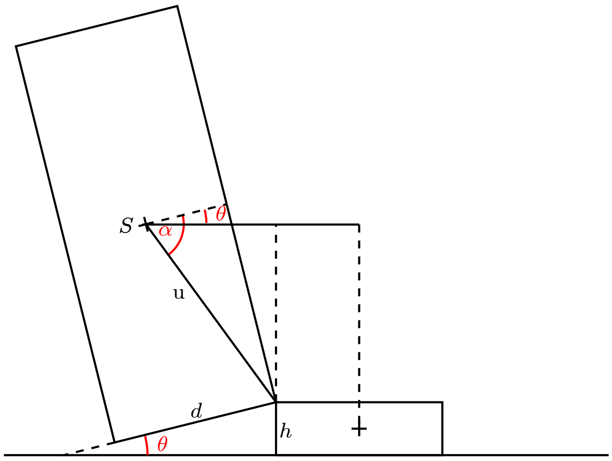


Figure 6: The bottom disc slides too far

And since $|\angle TSV| = \alpha - \theta$, it can be determined from the triangle STV that the distance between the centers of mass in this situation equals.

$$\Delta s = \frac{d}{2} + u \cdot \cos(\alpha - \theta)$$

Therefore whenever $\Delta s \geq \frac{d}{2} + u \cdot \cos(\alpha - \theta)$ we say the two bodies do not touch.

4.4 Free fall

When the first phase has terminated by distance, the tower does not start touching the ground immediately after. It must first fall down.

As we know the height of the lowest point of the cylinder is $h - d \cdot \sin\theta$, we can calculate the time of the free fall as:

$$h - d \cdot \sin\theta = \frac{g \cdot t_{\mathrm{f}}^2}{2}$$

$$t_{\mathrm{f}} = \sqrt{\frac{2(h - d \cdot \sin\theta)}{g}}$$

This is the time it will take the cylinder to lay on the ground[4]. During this it will continue to rotate. Therefore the angle will increase by:

$$\dot{\theta}\sqrt{\frac{2(h - d \cdot \sin\theta)}{g}}$$

4.5 Tower stabilization

When the cylinder slides from the bottom disc, the angular velocity changes (we will show this in the next subsection), but it still continues to rotate due to the angular momentum conservation law.

[4]Note that the actual height depends on the angle when the cylinder drops down. For simplicity we will stick to this analytical formula in this document, however in computational model we solved this numerically.

The friction force will not act on the cylinder anymore as the two objects do not touch, but both the gravity force and the reaction to it will continue to play a role.

As the cylinder now rotates around the base, the reaction force will have zero torque. However the gravity force acts in the center of mass and its torque therefore would not generally be zero.

The distance between the center and the point where the tower touches the ground is $u = \frac{\sqrt{L^2 + d^2}}{2}$. As the distance and the force are not perpendicular, we have to calculate the angle between these.

Using the angle β, where as seen in the picture $\beta = \arctan \frac{d}{L}$, we can calculate the angle between the force and the distance as $\theta - \beta$. Therefore the torque caused by gravity force will equal:

$$\tau = M \cdot g \cdot u \cdot \sin(\theta - \beta)$$

We can link the angular acceleration to the torque in a similar fashion to the case when the two bodies were touching.

$$\ddot{\theta} = \frac{\tau}{I}$$

Where the moment of inertia of turning the cylinder over its base equals[1]:

$$I = M \left(\frac{d^2}{16} + \frac{L^2}{3} \right)$$

Therefore the angular acceleration will equal:

$$\ddot{\theta} = \frac{g \cdot \sqrt{L^2 + d^2} \cdot \sin\left(\theta - \arctan \frac{d}{L}\right)}{\frac{d^2}{8} + \frac{2L^2}{3}}$$

4.6 Linking the angle and angular velocity between phases

The angle θ will remain the same (as could be seen from the figure 5).

However the angular velocity of these will change accordingly to the angular momentum conservation law. (We will denote the angular momentum and velocity around the center of mass as L_c respectively $\dot{\theta}_c$).

$$L_c = L$$

$$I_c \cdot \dot{\omega}_c = I \cdot \dot{\theta}$$

Using the moments of inertia stated in previous sections:

$$\dot{\theta} = \frac{\frac{d^2}{16} + \frac{L^2}{12}}{\frac{d^2}{16} + \frac{L^2}{3}} \dot{\theta}_c$$

Therefore the angular velocity will be roughly one fourth of the angular velocity before it had just before the disc and the cylinder stopped touching.

4.7 Process termination

So far we have only dealt with the discs glued together and have not split the cylinder yet as promised.

We can do this when we can imagine how the latter is influenced by the discs being independent on each other.

When the angle between the discs and the ground is tiny, the friction prevents the blocks from moving, therefore they act exactly as glued together.

This however changes when the angles is high enough and the discs start sliding which causes the whole tower to fall.

We therefore have to determine at which angle it changes from the first case to the second.

That is when the gravity pulling it down equals the maximum possible friction pulling it up.

$$m \cdot g \cdot \sin\theta = \mu_{\mathrm{k}} \cdot m \cdot g \cdot \cos\theta$$

$$\theta = \arctan \mu_{\mathrm{k}}$$

Whenever the angle is higher than the arctangent we say that the tower falls.

The tower is standing whenever the angle between it and the ground equals zero and the tower does not touch the disc anymore.

It can be shown that one of these happens (or the tower may be moving as a whole). When the disc are on the ground the gravity force will pushing to one of the sides, if it pushes it clockwise, it will eventually hit the ground, if clockwise, it will reach the critical angle for the discs to slide out[5].

4.8 Model summary

In previous sections we showed how is it possible to model the behaviour of the tower under simplified conditions — we treated the tower as the discs except for the bottom one were glued together.

We showed that the distance travelled by the bottom disc can be described by the following differential equation:

$$\ddot{s}_{\mathrm{d}}(t, s_{\mathrm{d}}) = \frac{F(t, s_{\mathrm{d}})}{m} - \mu_k^* \cdot n \cdot g - \mu_k \cdot (n-1) \cdot g$$

For the forces in our experiment we also derived the analytical solution of this equation.

The horizontal distance travelled by the center of mass of the cylinder equals:

$$s(t) = \frac{1}{2}\mu_{\mathrm{k}} \cdot g \cdot t^2$$

Using these, we can formulate the equation for the angular rotation of the cylinder:

$$\ddot{\theta} = \frac{12((\Delta s - \frac{d}{2})g + \left|\Delta s - \frac{d}{2}\right| \cot\left(\arcsin\left(\frac{(\Delta s - \frac{d}{2})\cos\theta}{\sqrt{\left(\frac{L}{2}\right)^2 + \left(\Delta s - \frac{d}{2}\right)^2 - L\left|\Delta s - \frac{d}{2}\right|\sin\theta}}\right)\right)\mu_{\mathrm{k}} g)}{\frac{3}{4}d^2 + L^2}$$

[5]This analysis was not completely precise a there exists another equilibrium — when the gravity force has zero torque and the angular velocity is zero. This corresponds to the cylinder "standing on its edge". However such equilibrium is unstable and therefore eventually it will fall to either one or the other side.

These equations describe the motion until

$$\Delta s \geq \frac{d}{2} + u \cdot \cos\alpha - \theta$$

or

$$\left(\Delta s - \frac{d}{2} \geq \frac{L}{2}\sin\theta \wedge \sin\theta \geq \frac{h}{\frac{d}{2}+x}\right) \vee \left(\Delta s - \frac{d}{2} \leq \frac{L}{2}\sin\theta \wedge \sin\theta \geq \frac{h}{\frac{d}{2}-x}\right)$$

If the phase terminated the second way, the angle increases by $\dot{\theta}\sqrt{\dfrac{2(h-d\cdot\sin\theta)}{g}}$.

As the cylinder drops to the ground, its angular velocity is slowed down:

$$\dot{\theta} = \frac{\frac{d^2}{16}+\frac{L^2}{12}}{\frac{d^2}{16}+\frac{L^2}{3}}\dot{\theta}_{\mathrm{c}}$$

Then the rotation can be described using the following differential equation:

$$\ddot{\theta} = \frac{g\cdot\sqrt{L^2+d^2}\cdot\sin\left(\theta - \arctan\frac{d}{L}\right)}{\frac{d^2}{2}+\frac{2L^2}{3}}$$

This equation describes the motion until $\theta = 0$ or $\theta = \arctan\mu_{\mathrm{k}}$.

When $\theta = 0$, the cylinder stands on its base and we can therefore say that the tower holds.

On the contrary, when $\theta = \arctan\mu_{\mathrm{k}}$, the discs slide down and we then say that the tower has fallen.

As these equations are not easily solvable analytically, we decided to solve them numerically. For the numerical solution we implemented the iterated Backward Euler method (as the standard Forward Euler method is rather error prone) in programming language Python.

5 Experiment

5.1 Hockey pucks

(The measurements and predictions for these experiments are found in the appendix.)

We made two different experiments with hockey pucks. The first one was intended to test our hypothesis about the dependency on the input force. We built a tower out of a few hockey pucks (measurements were made for 3, 4 and 5 pucks) and attached the bottom puck to a weight over a trolley, so we could know the applied force ($F = m_{\mathrm{w}} \cdot g$).

The results were pretty much in accordance with the qualitative model: with a small force, the tower moved as a whole, with a bigger force, it fell, and with an even bigger force, it remained standing. The threshold forces increased with the number of pucks. Our theoretical model predicted similar results, except the thresholds were noticeably lower. This can be explained by imperfections in the measurement – the tower wasn't perfectly straight because of human error (we will return to it later) and the force may have fluctuated due to the method of applying it. However, the overall trend does match. There was also an interesting oddity in the predictions: for a certain combination of F and n, it showed a standing tower among otherwise falling towers. This is presumably caused by how the model accounts for when the cylinder touches the ground while still touching the bottom disc. This allows the tower to remain standing under rare conditions such as this. We however were not able to model these conditions in our experiments to determine whether such tower would remain standing.

We also used the same technique with the same pucks to examine the dependency on friction. We made a control measurement under normal conditions, then covered the pucks with tape and then rough paper to decrease/increase the friction respectively. As predicted by our qualitative model, lower friction made it easier to pull out the bottom puck without the tower collapsing; however, we didn't do any quantitative comparison because trying to measure the different friction coefficients would have a too large error margin.

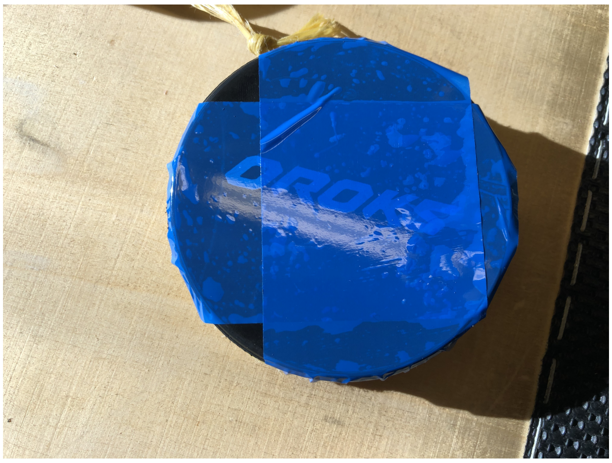


Figure 7: Taped hockey puck

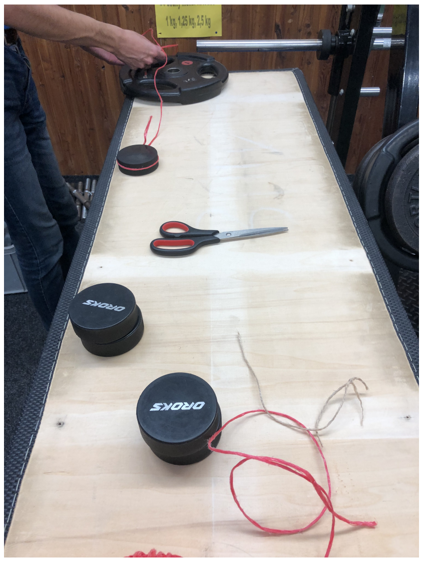


Figure 8: Experimental setup

5.2 Game tokens

We built a simple contraption that used force measuring springs to shoot out the bottom token from a tower of small tokens from game *Kde leží Uppsala*. The contraption comprised of two force measuring springs attached to both ends of a metal ruler, this plate was was very thin. We then expanded the strings until they showed desired force, placed the tower in front of them and released the ruler. We tried to release the ruler as close to the tower as possible so the ruler didn't gain any speed before contact. We used this contraption to make many measurements concerning the dependency of success on the number of discs and force.

$$d = 15.0\,\text{mm} \pm 0.5\,\text{mm},\ h = 4.0\,\text{mm} \pm 0.5\,\text{mm},\ m = 0.55\,\text{g} \pm 0.05\,\text{g}$$

The force measuring springs had a force of 0.2 N per centimeter of expansion. For example, if the force pulling the token had 7 N and the disc had a diameter of 1 cm, the force would have a linear decrease from 7 N to 6.6 N. This is the case because two springs collapse by 1 cm thus decreasing the force by $1 \cdot 0.2 \cdot 2$ Newtons.

In all the experiments all the towers were mostly stable for all the forces when they had less than around 16 tokens. When more tokens were added, the tower started falling.

However our theory does not predict this. According to the theory the tower should be standing even with many more discs. For reference, Figure 9 shows the predicted results (yellow = remains standing, blue = falls, violet = slides). It can be seen that we had to plot significantly smaller forces in order for the model to predict that the tower would fall.

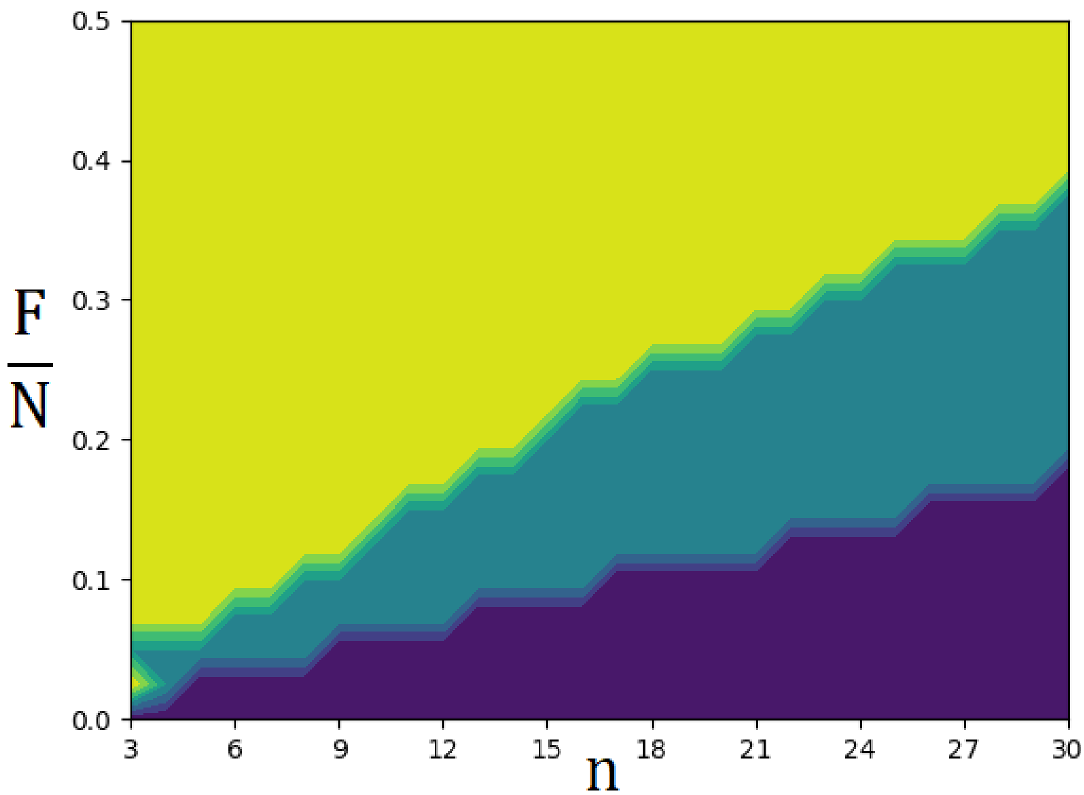


Figure 9: The model predictions for tokens

In order to find out why our theory does not match the experiment, we focused on the parameters which varied from those of the hockey pucks — there our theory matched the experiment reasonably well.

The first parameter that differed significantly was the friction coefficient — the coefficient for pucks was much higher than that for tokens. Due to this the pucks stacked together better. It could have been that for smaller coefficients the approximation to a cylinder was unsuitable.

In order to examine this we repeated the experiment with the same tokens, however now the tokens were glued together by a sellotape.

The tower remained standing when there were less than 14 – 18 tokens, then it once again started falling. Therefore the problem was not with the cylinder approximation.

Our another assumption was that the discs were stacked precisely on the top of the disc below. However, any tower built by a human will not fulfill these conditions. If the inaccuracies in laying the discs were the cause why the theory did not predict the behaviour correctly, it could also explain why the pucks were affected less than the game tokens.

When a person builds a tower, the inaccuracies are usually always the same as they are mostly caused by the fact that at very tiny distances, it is hard to distinguish if there are any shifts in the horizontal position between the two pucks. We can denote the maximal shift between two consecutive pucks as e. Then the relative shift, which should be the relevant factor, can be computed as:

$$\epsilon = \frac{e}{d}$$

Therefore the extent to which the tower is effected by this decreases with increasing diameter of the disc.

In order to verify this, we tried to artificially increase the value of e — we built the tower in a way such that the deviations in positions were still small, yet visible. After this the tower started falling at ten tokens. When we increased the deviations even more the tower did not remain standing even when there were only five discs[6].

It may seem that at this point, the theory becomes useless, as the whole process is chaotic (e.i. the result varies largely with small deviations in the input) and our model cannot predict this. This is not entirely true as we could calculate the probability of the tower standing by generating many possible setups and then computing the output for these.

In order to do this all the equation would have to be extended an arbitrary shape than just cylinder and as this would very much complicate the math behind, we did not do this.

5.2.1 Random walk

In the experiments with game tokens, there was a critical value of height and above this the tower always fell down.

We should therefore explain why this happen and why the increasing number of discs makes the tower to be more prone to falling.

Although it may seem intuitive that with more tokens the upper most disc should have higher distance, proving this is rather complicated and out of the scope of this document. We will therefore only stick to an estimation of this.

Lets assume all the distances between centers of two consecutive discs is about e (the error due to the human imperfections).

In order to make the estimation we will only investigate the one dimension case. We will always add disc in a way that it is left or right by e from the disc before.

This process is called the random walk and it can be shown[2] that with n discs the top

[6] These experiments were intended to be only qualitative therefore were not measured for all the values, as it would be hard to define the deviation exactly. The used force was 1N

most will be away from the bottom one on average in order of $e \cdot \sqrt{n}$.

From this we can see that the distance from the bottom disc will increase with more discs and this makes the tower more and more unstable.

6 Conclusion

We saw that whether the tower remains standing or not depends on two types of parameters.

The first class of parameters are the controllable — the applied force, the number of discs and their parameters. Our theory dealt primarily with these.

There is however a second class of parameters, those which are not controllable — these are the deviations between the horizontal positions of the consecutive discs. Although these have an upper bound (whose value is not known to us as it depends on the human's inaccuracy), they are essentially random. The randomness in these is what makes the system to behave rather chaotically.

As these second parameters were neglected in our theory, its predictions differed from the results of our experiments.

Due to the inaccuracies in the tower, which caused it to be less stable, our model always predicted lower forces to be needed for the tower to hold. The errors of these however tend to decrease with increasing diameter of the disk — in our experiments with game tokens the error was large, in the experiment with small hockey pucks it was smaller and in the experiment with big hockey pucks the theory roughly predicted the experiment.

We also showed why there is such relationship as we are only interested in the relative error of the deviations. And this one decreases with increasing diameter.

It could be that for large enough diameters of discs our theory would produce exact prediction, however we did not prove this experimentally. Nevertheless, for the diameters we used in our experiments, a randomized version of our theory including the random deviations would be more suitable.

On the other hand, our theory predicted the nature of the controllable parameters correctly.

We showed in the qualitative section that with increasing force the tower should fall less.

Our model predicted this as well and our experiments with hockey pucks proved this behaviour.

The theoretical model also predicted that with increasing number of discs, the tower should be more prone to falling. In all the experiments this was true, however it is hard to estimate to which extent this was affected by the random deviations.

Another correctly predicted behaviour was the dependency on the friction coefficient. Although from the qualitative analysis it was unsure how does the friction coefficient affects the chance of falling. On one hand the higher coefficients made the discs rotate more which is the cause of falling. On the other hand the disc could slide easily which may cause the fall of tower when stabilizing. The quantitative model predicted that the rotation had bigger impact and that with increasing friction coefficient the tower should be more likely to fall. This was verified by our experiments with hockey pucks with different friction coefficients.

Over all our qualitative and quantitative analysis provided an insight into how does the process depend on various controllable parameters. Our experiments followed the same trends, but they showed that the behaviour is not so easily determinable and that it depends on random deviations. Later we discussed how is it possible to lower the impact of these with increasing disc sizes.

References

[1] Georgia State University, Moment of Inertia: Cylinder, http://hyperphysics.phy-astr.gsu.edu/hbase/icyl.html

[2] Wolfram Alpha, the random walk, https://mathworld.wolfram.com/RandomWalk1-Dimensional.html

A Geometry details with friction torque

Since the angles $\angle SUT$ and $\angle SVT$ are always right angles, all the points $STUV$ lie on one circle. The angle $\angle UTV$ clearly equals $\frac{\pi}{2} - \theta$ in the first two cases and $\frac{\pi}{2} - \theta$ in the third case, so according to the inscribed angle theorem, $|\angle USV|$ equals $\frac{\pi}{2} - \theta$ in the first and third case and $\frac{\pi}{2} + \theta$ in the second case. From the law of cosines, we can determine that $p = |UV| = \sqrt{|SU|^2 + |SV|^2 - 2|SU||SV| \cos |\angle USV|}$. Applying a trigonometric identity to the last term, it equals $\pm 2|SU||SV| \sin \theta$, with the $\pm$ depending on the case. However, since $\Delta S - \frac{d}{2}$ happens to be negative in and only in the second case, we can succintly write this as $-2\frac{l}{2}\left|\Delta S - \frac{d}{2}\right| \sin \theta$. Then we can use the sine theorem to determine the angles $\angle SVU$ and $\angle SUV$, which by the inscribed angle theorem equal $\angle STU$ and $\angle STV$ respectively. (In the first case, it's actually $|\angle STU| = \pi - |\angle SVU|$, but that doesn't matter since we're taking the arcsine anyway, which always produces the acute angle (and $\angle STU$ will always be acute; the proof of it is out of scope of this document)).

B Large hockey pucks – data

The following are data for big hockey pucks. We always did 3 measurements with different forces for 3,4 or 5 pucks stacked together. *fall* means that tower fell, *remain* that the lowest puck was removed and the rest remained intact and *moving* that the whole tower was moving in the direction of force.

$$d = 75.0\,\mathrm{mm} \pm 0.5\,\mathrm{mm},\ h = 25.0\,\mathrm{mm} \pm 0.5\,\mathrm{mm},\ m = 171\,\mathrm{g} \pm 0.05\,\mathrm{g}$$

B.1 3 pucks

$\frac{F}{N}$	1. measurement	2. measurement	3. measurement
25	moving	moving	fall
38	fall	fall	fall
45	remain	fall	fall
50	fall	remain	remain
63	remain	remain	remain

B.2 4 pucks

$\frac{F}{N}$	1. measurement	2. measurement	3. measurement
25	moving	fall	fall
38	fall	remain	fall
50	fall	fall	fall
55	fall	remain	remain
63	fall	remain	remain

B.3 5 pucks

$\frac{F}{N}$	1. measurement	2. measurement	3. measurement
25	fall	fall	fall
38	fall	fall	fall
50	remain	fall	fall
63	fall	fall	fall
75	fall	fall	fall

B.4 Theoretical predictions

We counted with $\mu_s = \mu_s^* = 1.0$ and $\mu_k = \mu_k^* = 0.8$. Yellow = tower remains standing, blue = tower falls, violet = tower moves. Note that in order to smoothen the plot, we allowed a fractional number of discs – this doesn't affect the model since it turns all but one disc into a monolith and counts with n as a continuous value.

bigpucks.png

C Small hockey pucks – data

The following are data for small hockey pucks. We always did 3 measurements with different forces for 3,4 or 5 pucks stacked together. *fall* means that tower fell, *remain* that the lowest puck was removed and the rest remained intact and *moving* that the whole tower was moving in the direction of force.

$$d = 60.0\,\mathrm{mm} \pm 0.5\,\mathrm{mm},\ h = 20.0\,\mathrm{mm} \pm 0.5\,\mathrm{mm},\ m = 84.7\,\mathrm{g} \pm 0.05\,\mathrm{g}$$

C.1 3 pucks

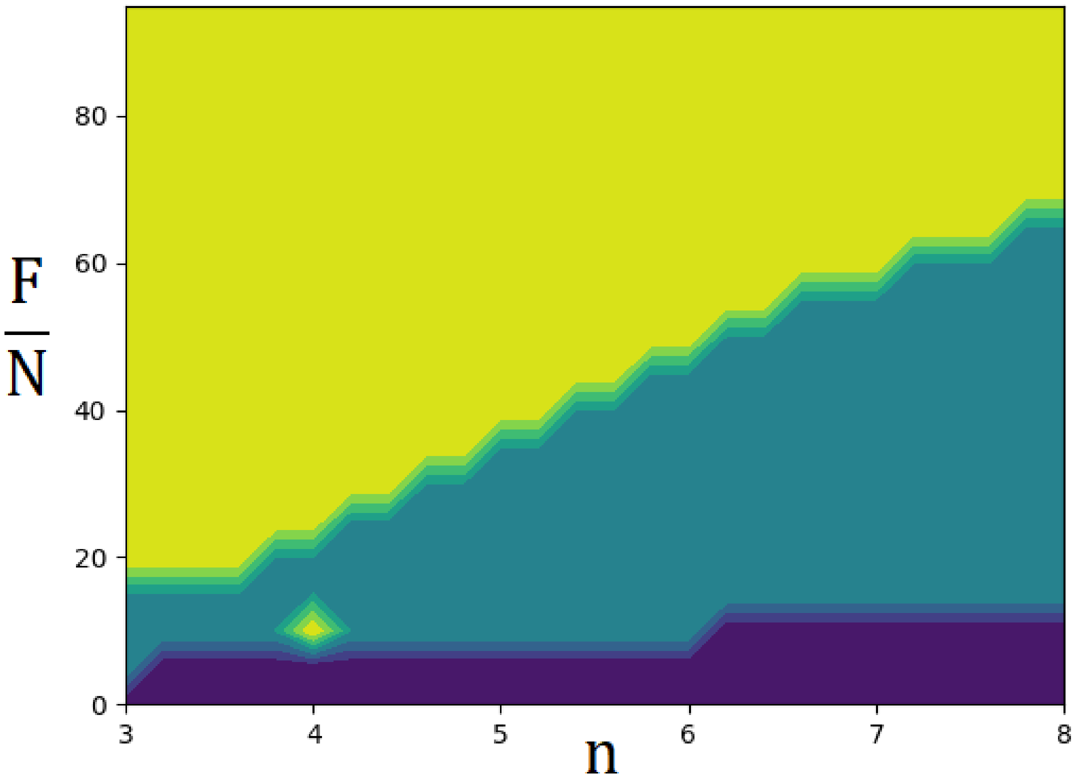
$\frac{F}{N}$	1. measurement	2. measurement	3. measurement
25	fall	moving	moving
38	moving	fall	moving
45	remain	remain	fall
50	remain	remain	remain

C.2 4 pucks

$\frac{F}{N}$	1. measurement	2. measurement	3. measurement
25	fall	fall	moving
38	moving	fall	fall
50	remain	remain	remain
63	remain	remain	remain

C.3 Theoretical predictions

We counted with $\mu_s = \mu_s^* = 1.0$ and $\mu_k = \mu_k^* = 0.8$. Yellow = tower remains standing, blue = tower falls, violet = tower moves. Note that in order to smoothen the plot, we allowed a fractional number of discs – this doesn't affect the model since it turns all but one disc into a monolith and counts with n as a continuous value.


F/N
n
0
20
40
60
80
3
4
5
6
7
8

D Big hockey pucks friction – data

The following are data for friction examined with big hockey pucks. We always did 4 measurements with $F = 75\text{N}$ pucks stacked together. *fall* means that tower fell, *remain* that the lowest puck was removed and the rest remained intact and *moving* that the whole tower was moving in the direction of force.

$$d = 75.0\,\text{mm} \pm 0.5\,\text{mm},\ h = 25.0\,\text{mm} \pm 0.5\,\text{mm},\ m = 171\,\text{g} \pm 0.05\,\text{g}$$

D.1 Control measurement

N of pucks	1. measurement	2. measurement	3. measurement	4. measurement
4	fall	remain	remain	fall
5	fall	fall	remain	remain

D.2 With tape between the two lowest

N of pucks	1. measurement	2. measurement	3. measurement	4. measurement
4	remain	remain	remain	remain
5	remain	remain	remain	remain

D.3 With tape between all the discs, contact with pad is normal

N of pucks	1. measurement	2. measurement	3. measurement	4. measurement
4	remain	remain	remain	remain
5	remain	fall	fall	remain

D.4 Thick paper, contact with pad is normal

N of pucks	1. measurement	2. measurement	3. measurement	4. measurement
4	fall	fall	remain	fall
5	fall	fall	remain	fall